AD-A115 355 STANFORD UNIV CA DEPT OF STATISTICS F/G 12/1

DISTRIBUTIONS OF QUADRATIC FORMS AND COCHRAN'S THEOREM FOR ELLI--ETC(U)

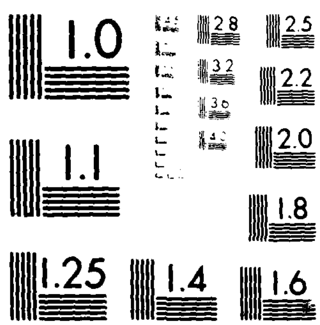
MAY 82 T W ANDERSON, K FANG N00014-75-C-0442

UNCLASSIFIED TR-53 NL

1 of 1

AD A
115355

END
DATE
FILMED
07-82
DTIC



MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS 1963 A

8

DISTRIBUTIONS OF QUADRATIC FORMS AND COCHRAN'S THEOREM FOR ELLIPTICALLY CONTOURED DISTRIBUTIONS AND THEIR APPLICATIONS

BY

T. W. ANDERSON and KAI-TAI FANG

TECHNICAL REPORT NO. 53
MAY 1982

PREPARED UNDER CONTRACT N00014-75-C-0442
(NR-042-034)
OFFICE OF NAVAL RESEARCH
THEODORE W. ANDERSON, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

DISTRIBUTIONS OF QUADRATIC FORMS AND COCHRAN'S THEOREM FOR ELLIPTICALLY CONTOURED DISTRIBUTIONS AND THEIR APPLICATIONS

T. W. Anderson
Stanford University

and

Kai-Tai Fang
Stanford University

and

Institute of Applied Mathematics
Academia Sinica, Beijing, China

TECHNICAL REPORT NO. 53

MAY 1982

PREPARED UNDER CONTRACT N00014-75-C-0442
(NR-042-034)
OFFICE OF NAVAL RESEARCH

Theodore W. Anderson, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

DISTRIBUTIONS OF QUADRATIC FORMS AND COCHRAN'S THEOREM FOR ELLIPTICALLY CONTOURED DISTRIBUTIONS AND THEIR APPLICATIONS

T. W. Anderson
Stanford University

and

Kai-Tai Fang
Stanford University

and

Institute of Applied Mathematics
Academia Sinica, Beijing, China

1. Introduction.

If the characteristic function of an n-dimensional random vector $\underset{\sim}{x}$ has the form $e^{i\underset{\sim}{t}'\underset{\sim}{\mu}}\phi(\underset{\sim}{t}'\underset{\sim}{\Sigma}\underset{\sim}{t})$, where $\underset{\sim}{\mu}:n\times 1$, $\underset{\sim}{\Sigma}:n\times n$, and $\underset{\sim}{\Sigma} \geq 0$, we say that $\underset{\sim}{x}$ is distributed according to an elliptically contoured distribution with parameters $\underset{\sim}{\mu}$, $\underset{\sim}{\Sigma}$ and ϕ, and we write $\underset{\sim}{x} \sim EC_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma},\underset{\sim}{\phi})$ (cf. Cambanis, Huang, and Simons (1981)). In particular, when $\underset{\sim}{\mu} = 0$ and $\underset{\sim}{\Sigma} = \underset{\sim}{I}_n$ (the identity matrix), $EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\underset{\sim}{\phi})$ is called a spherical distribution (cf. Kelker (1970)).

The class of the elliptically contoured distributions contains such distributions as the multivariate normal distribution, the multivariate t-distribution, the multivariate Cauchy distribution, the multivariate Laplace distribution and the multivariate uniform distribution on the sphere in $\mathbb{R}^n$ or in the sphere in $\mathbb{R}^n$.

The theory of elliptically contoured distributions has been discussed by Schoenberg (1938), Lord (1954), Kelker (1970), Das Gupta et al. (1972), Devlin, Gnanadesikan and Kettenring (1976), Kariya and Eaton (1977), Muirhead (1980) and Cambanis, Huang, and Simons (1981). The following basic properties obtained by the above authors are needed in this paper.

(1) Let $\underset{\sim}{u}^{(n)}$ denote a random vector which is uniformly distributed on the unit sphere in $\mathbb{R}^n$ and $\Omega_n(\|\underset{\sim}{t}\|^2)$ denote its characteristic function. If Φ_n, $n \geq 1$, is the class of all functions $\phi:[0,\infty) \to \mathbb{R}$ such that $\phi(\|\underset{\sim}{t}\|^2)$ is a characteristic function, then $\phi \in \Phi_n$ if and only if

$$(1.1) \qquad \phi(t) = \int_0^\infty \Omega_n(r^2 t)\,dF(r), \qquad t \geq 0,$$

for some distribution function F on $[0,\infty)$ (cf. Schoenberg (1938)).

(2) $\underset{\sim}{x} \sim EC_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma},\phi)$ with rank $rk(\Sigma) = k$ if and only if

$$(1.2) \qquad \underset{\sim}{x} \overset{d}{=} \underset{\sim}{\mu} + R\underset{\sim}{A}'\underset{\sim}{u}^{(k)}$$

where $R \geq 0$ is independent of $\underset{\sim}{u}^{(k)}$, $\underset{\sim}{\Sigma} = \underset{\sim}{A}'\underset{\sim}{A}$ is a factorization of $\underset{\sim}{\Sigma}$ (i.e., $\underset{\sim}{A}$ is a $k \times n$ matrix and $rk(\underset{\sim}{A}) = k$) and the distribution function $F(x)$ of R is related to ϕ as in (1.1) with k substituted for n. Here, $\underset{\sim}{x} \overset{d}{=} \underset{\sim}{y}$ denotes that the random vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are identically distributed (cf. Cambanis, Huang, and Simons (1981)). In the next section we discuss the operation "$\overset{d}{=}$" in detail.

(3) If $\underset{\sim}{x} \sim N_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$, then $\underset{\sim}{x} \sim EC_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma},\phi)$ with $\phi(t) = \exp(-t/2)$ and $R^2 \sim \chi^2_n$, where χ^2_n denotes the chi-squared distribution with n degrees of freedom.

If $\underset{\sim}{A}$ is an $n \times n$ symmetric matrix, what is the distribution of $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$? Kelker (1970) obtained the distribution of $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}/\underset{\sim}{x}'\underset{\sim}{x}$ in the case of $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0}, \underset{\sim}{I}_n, \phi)$ and the distribution of $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ under the condition that $\underset{\sim}{x}$ has finite fourth moments and $\underset{\sim}{x}$ has a density. Kariya and Eaton (1977) gave the distributions of $\underset{\sim}{a}'\underset{\sim}{x}/\|\underset{\sim}{x}\|$ and $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}/\|\underset{\sim}{x}\|^2$, where $\|\underset{\sim}{x}\|^2 = \underset{\sim}{x}'\underset{\sim}{x}$. There is a rich bibliography on the distribution of quadratic forms in normal population. Recently, Anderson and Styan (1980) reviewed various extensions of Cochran's Theorem in a bibliographic and historical setting. Our purpose in this paper is to extend Cochran's Theorem to elliptically contoured distributions in various aspects. The main results are in Sections 4 and 5. In Section 3 we list some basic properties of Dirichlet distributions that are needed later. Some applications are given in Section 6.

Throughout the paper $N_n(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ denotes the n-dimensional normal distribution with mean $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{\Sigma}$, χ_k^2 denotes the chi-squared distribution with k degrees of freedom, $F(k,\ell)$ denotes the F-distribution with k and ℓ degrees of freedom, t_n denotes the t-distribution with n degrees of freedom, $\underset{\sim}{I}_n$ denotes the $n \times n$ identity matrix, $\underset{\sim}{\mathcal{E}}_n$ denotes the $n \times 1$ vector with elements 1, $rk(\underset{\sim}{A})$ denotes the rank of the matrix $\underset{\sim}{A}$, and $\underset{\sim}{A}^-$ denotes a generalized inverse.

2. The Operation "$\overset{d}{=}$".

If random vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ have the same distribution, we denote that fact by $\underset{\sim}{x} \overset{d}{=} \underset{\sim}{y}$.

(1) Assume $\underset{\sim}{x}$, $\underset{\sim}{y}$, $\underset{\sim}{z}$ and $\underset{\sim}{w}$ are random vectors, $\underset{\sim}{x}$ and $\underset{\sim}{z}$ are independent, $\underset{\sim}{y}$ and $\underset{\sim}{w}$ are independent, and $x \overset{d}{=} y$. Then $z \overset{d}{=} w$ if and only if

$$\underset{\sim}{x} + \underset{\sim}{z} \overset{d}{=} \underset{\sim}{y} + \underset{\sim}{w}. \tag{2.1}$$

In particular, if $\underset{\sim}{z}$ is a nonrandom vector, then $\underset{\sim}{x} = \underset{\sim}{y}$ implies $\underset{\sim}{x} + \underset{\sim}{z} \overset{d}{=} \underset{\sim}{y} + \underset{\sim}{z}$.

(2) Assume that $\underset{\sim}{x} \overset{d}{=} \underset{\sim}{y}$ and $f_j(\cdot)$, $j = 1,\ldots,m$, are Borel functions, then

$$(2.2) \qquad \begin{pmatrix} f_1(\underset{\sim}{x}) \\ f_2(\underset{\sim}{x}) \\ \vdots \\ f_m(\underset{\sim}{x}) \end{pmatrix} \overset{d}{=} \begin{pmatrix} f_1(\underset{\sim}{y}) \\ f_2(\underset{\sim}{y}) \\ \vdots \\ f_m(\underset{\sim}{y}) \end{pmatrix} .$$

For instance, we have

$$(2.3) \qquad \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \overset{d}{=} \underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}, \quad \underset{\sim}{x}'\underset{\sim}{x} \overset{d}{=} \underset{\sim}{y}'\underset{\sim}{y},$$

$$(2.4) \qquad \begin{pmatrix} \underset{\sim}{x}'\underset{\sim}{A}_1\underset{\sim}{x} \\ \vdots \\ \underset{\sim}{x}'\underset{\sim}{A}_m\underset{\sim}{x} \end{pmatrix} \overset{d}{=} \begin{pmatrix} \underset{\sim}{y}'\underset{\sim}{A}_1\underset{\sim}{y} \\ \\ \underset{\sim}{y}'\underset{\sim}{A}_m\underset{\sim}{y} \end{pmatrix} .$$

(3) The following fact is important in this paper.

<u>Lemma 1</u>. Assume that $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are $n \times 1$ random vectors, z is a random variable and is independent of $\underset{\sim}{x}$ and $\underset{\sim}{y}$, respectively. If

$$(2.5) \qquad P(z > 0) = 1$$

then $\underset{\sim}{x} \overset{d}{=} \underset{\sim}{y}$ if and only if $z\underset{\sim}{x} \overset{d}{=} z\underset{\sim}{y}$.

<u>Proof</u>. Firstly, we prove that Lemma 1 holds if

$$(2.6) \qquad P(x_i > 0) = P(y_i > 0) = 1, \quad i = 1,\ldots,n .$$

By using (2.2) we have

$$\underset{\sim}{zx} \overset{d}{=} \underset{\sim}{zy} \Leftrightarrow \begin{pmatrix} \ln zx_1 \\ \vdots \\ \ln zx_n \end{pmatrix} \overset{d}{=} \begin{pmatrix} \ln zy_1 \\ \vdots \\ \ln zy_n \end{pmatrix} \Leftrightarrow \ln z \underset{\sim}{\varepsilon}_n + \begin{pmatrix} \ln x_1 \\ \vdots \\ \ln x_n \end{pmatrix} \overset{d}{=} \ln z \underset{\sim}{\varepsilon}_n + \begin{pmatrix} \ln y_1 \\ \vdots \\ \ln y_n \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \ln x_1 \\ \vdots \\ \ln x_n \end{pmatrix} \overset{d}{=} \begin{pmatrix} \ln y_1 \\ \vdots \\ \ln y_n \end{pmatrix} \Leftrightarrow \underset{\sim}{x} \overset{d}{=} \underset{\sim}{y} \ .$$

Secondly, we consider $\underset{\sim}{x}$ and $\underset{\sim}{y}$ to be arbitrary random vectors. It is enough to prove the lemma in the case of $\underset{\sim}{x}$ and $\underset{\sim}{y}$ being scalar random variables because

$$\underset{\sim}{x} \overset{d}{=} \underset{\sim}{y} \Leftrightarrow \underset{\sim}{a}'\underset{\sim}{x} \overset{d}{=} \underset{\sim}{a}'\underset{\sim}{y}, \quad \forall \underset{\sim}{a} \in \mathbb{R}_n \Leftrightarrow z\underset{\sim}{a}'\underset{\sim}{x} \overset{d}{=} z\underset{\sim}{a}'\underset{\sim}{y}, \quad \forall \underset{\sim}{a} \in \mathbb{R}_n$$

$$\Leftrightarrow \underset{\sim}{a}'(z\underset{\sim}{x}) \overset{d}{=} \underset{\sim}{a}'(z\underset{\sim}{y}), \quad \forall \underset{\sim}{a} \in \mathbb{R}_n \Leftrightarrow z\underset{\sim}{x} \overset{d}{=} z\underset{\sim}{y}$$

if we have proved the lemma in the above case. Let

$$f_+(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}, \quad f_-(x) = \begin{cases} 0 & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

and $x^+ = f_+(x)$, $x^- = f_-(x)$; $y^+ = f_+(y)$ and $y^- = f_-(y)$. It is easy to verify that

$$x \overset{d}{=} y \Leftrightarrow x^+ \overset{d}{=} y^+, \ x^- \overset{d}{=} y^-, \text{ and } p(x=0) = p(y=0) \Leftrightarrow x^+ \overset{d}{=} y^+ \text{ and } x^- \overset{d}{=} y^- \ .$$

If we can prove that $zx^+ \overset{d}{=} zy^+ \Leftrightarrow x^+ \overset{d}{=} y^+$ and $zx^- \overset{d}{=} zy^- \Leftrightarrow x^- \overset{d}{=} y^-$, then the assertion follows from

$$zx \stackrel{d}{=} zy \Leftrightarrow (zx)^+ \stackrel{d}{=} (zy)^+ \quad \text{and} \quad (zx)^- \stackrel{d}{=} (zy)^- \Leftrightarrow$$

$$zx^+ \stackrel{d}{=} zy^+ \quad \text{and} \quad zx^- \stackrel{d}{=} zy^- \Leftrightarrow x^+ \stackrel{d}{=} y^+ \quad \text{and} \quad x^- \stackrel{d}{=} y^- \Leftrightarrow x \stackrel{d}{=} y\,.$$

Now we prove that $zx^+ \stackrel{d}{=} zy^+ \Leftrightarrow x^+ \stackrel{d}{=} y^+$. Assume $zx^+ \stackrel{d}{=} zy^+$, then

$$p(x>0) = p(x^+>0) = p(zx^+>0) = p(zy^+>0) = p(y^+>0) = p(y>0)$$

and

$$p(x>0)E(e^{itzx}|x>0) = E(e^{itzx^+}) - p(x\le 0) = E(e^{itzy^+}) - p(y\le 0)$$

$$= E(e^{itzy}|y>0)\; p(y>0)\,.$$

From the first part of the proof we have $E(e^{itx}|x>0) = E(e^{ity}|y>0)$, i.e., $x^+ \stackrel{d}{=} y^+$. The "$\Leftarrow$" part follows by the same technique. Similarly we have $zx^- \stackrel{d}{=} zy^- \Leftrightarrow x^- \stackrel{d}{=} y^-$. Q.E.D.

3. The Dirichlet Distribution.

If $\underset{\sim}{y} = (y_1,\ldots,y_m)'$ is a random vector with $\Sigma_1^m y_i = 1$ and $(y_1,\ldots,y_{m-1})'$ has the density with $\alpha_i > 0$, $i = 1,\ldots,m$,

$$(3.1) \qquad p_m(t_1,\ldots,t_{m-1}) = \frac{\Gamma\left(\sum_1^m \alpha_i\right)}{\prod_1^m \Gamma(\alpha_i)} \prod_1^{m-1} t_i^{\alpha_i-1}\left(1 - \sum_1^{m-1} t_i\right)^{\alpha_m-1}$$

$$\text{if } t_i \ge 0\,, \quad i = 1,\ldots,m-1\,, \quad \sum_1^{m-1} t_i < 1\,,$$

$$= 0\,, \qquad \text{otherwise.}$$

We say that $\underset{\sim}{y}$ is distributed according to a Dirichlet distribution and denote it as $(y_1,\ldots,y_{m-1}) \sim D_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m)$. When $m=2$ $D_2(d_1;d_2)$ reduces to the Beta distribution, $B(\alpha_1,\alpha_2)$.

It is a well-known fact that if $\underset{\sim}{x} \sim N_n(\underset{\sim}{0},\underset{\sim}{I}_n)$ and is partitioned into m parts $\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(m)}$ with $\alpha_1,\ldots,\alpha_m$ components of $\underset{\sim}{x}$, respectively, i.e.,

$$(3.2)\qquad \underset{\sim}{x} = \begin{pmatrix} \underset{\sim}{x}_{(1)} \\ \vdots \\ \underset{\sim}{x}_{(m)} \end{pmatrix} ,$$

then the joint density of $\left(\frac{\underset{\sim}{x}'_{(1)}\underset{\sim}{x}_{(1)}}{\|\underset{\sim}{x}\|^2},\ldots,\frac{\underset{\sim}{x}'_{(m-1)}\underset{\sim}{x}_{(m-1)}}{\|\underset{\sim}{x}\|^2}\right)$ is $D_m\left(\frac{\alpha_1}{2},\ldots,\frac{\alpha_{m-1}}{2};\frac{\alpha_m}{2}\right)$.

In particular, if $\alpha_1 = \ldots = \alpha_k = 1$, $\alpha_{k+1} = n-k$, $m = k+1$, then $\left(\frac{x_1^2}{\|\underset{\sim}{x}\|^2},\ldots,\frac{x_k^2}{\|\underset{\sim}{x}\|^2}\right)$ has $D_{k+1}\left(\frac{1}{2},\ldots,\frac{1}{2};\ \frac{n-k}{2}\right)$ as its density. It can be verified that the density of $\left(\frac{|x_1|}{\|\underset{\sim}{x}\|},\ldots,\frac{|x_k|}{\|\underset{\sim}{x}\|}\right)$ is

$$(3.3)\qquad \frac{\Gamma(\frac{n}{2})\ 2^k}{\Gamma(\frac{n-k}{2})\pi^{\frac{1}{2}k}} \left(1 - \sum_1^k x_i^2\right)^{\frac{n-k}{2}-1} \qquad \text{if } x_i \geq 0,\quad i=1,\ldots,k\ ;\ \sum_1^k x_i^2 < 1 ,$$

and the density of $\left(\frac{x_1}{\|\underset{\sim}{x}\|},\ldots,\frac{x_k}{\|\underset{\sim}{x}\|}\right)$ is

$$(3.4)\qquad g_n(x_1,\ldots,x_k) \equiv \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-k}{2})\pi^{k/2}} \left(1 - \sum_1^k x_i^2\right)^{\frac{n-k}{2}-1} ,\ \text{if } \sum_1^k x_i^2 < 1 .$$

If $\underset{\sim}{x} \sim E\,C_n(\underset{\sim}{0},\underset{\sim}{I}_n\phi)$, then (cf. (1.2))

$$(3.5) \qquad \underset{\sim}{x} \overset{d}{=} R\underset{\sim}{u}^{(n)} ,$$

where R and $\underset{\sim}{u}^{(n)}$ are independent. From (2.3)

$$(3.6) \qquad \|\underset{\sim}{x}\|^2 = \underset{\sim}{x}'\underset{\sim}{x} \overset{d}{=} R^2\underset{\sim}{u}^{(n)\prime}\underset{\sim}{u}^{(n)} = R^2 \quad \text{or} \quad \|\underset{\sim}{x}\| \overset{d}{=} R .$$

If $p(\underset{\sim}{x}=\underset{\sim}{0}) = 0$ (or $p(R=0)=0$), we have (cf. Cambanis, Huang, and Simons, (1981), Lemma 1)

$$(3.7) \qquad \underset{\sim}{x}/\|\underset{\sim}{x}\| \overset{d}{=} \underset{\sim}{u}^{(n)} ,$$

and it is independent of $\|\underset{\sim}{x}\| \overset{d}{=} R$. Note that (3.7) holds for all of the class of spherical distributions with $p(\underset{\sim}{x}=\underset{\sim}{0}) = 0$. Thus we can assume $\underset{\sim}{x} \sim N_n(\underset{\sim}{0},\underset{\sim}{I}_n)$ to obtain the distribution of ratios. We will use the property many times. Denote $\underset{\sim}{u}^{(n)} = (u_1,\ldots,u_n)'$. We immediately obtain that

(i) the density of $u_{i_1},\ldots,u_{i_k}$, $1 \le i_1 < i_2 < \cdots < i_k \le n$ is $g_n(x_1,\ldots,x_k)$, (cf. (3.4));

(ii) the density of $|u_{i_1}|,\ldots,|u_{i_k}|$, $1 \le i_1 < i_2 < \cdots < i_k \le n$, is (3.3);

(iii) if $\underset{\sim}{u}^{(n)}$ is partitioned into m parts as the same as (3.2), then the density of $(\|\underset{\sim}{u}_{(1)}\|^2,\ldots,\|\underset{\sim}{u}^{(2)}_{(m-1)}\|)$ is $D_m(\frac{1}{2}\alpha_1,\ldots,\frac{1}{2}\alpha_{m-1}; \frac{1}{2}\alpha_m)$.

(iv) Let $\mu_{r_1,\ldots,r_m} = E[\prod_1^m u_i^{r_i}]$, then $\mu_{r_1,\ldots,r_m} = 0$ if some r_i are odd. If all of $r_1,\ldots,r_m$ are even, $\mu_{r_1,\ldots,r_m}$ can be obtained as moments of the Dirichlet distribution (cf. Johnson and Kotz (1972)), i.e.

$$(3.8) \qquad \mu_{r_1,\ldots,r_m} = \frac{\prod_1^m [\frac{1}{2}]^{[r_j]}}{[\frac{n}{2}]^{[\sum_1^m r_j]}} ,$$

where $x^{[k]} = x(x+1)\cdots(x+k-1)$. In particular, $E\underset{\sim}{u}^{(n)} = \underset{\sim}{0}$, $E\underset{\sim}{u}^{(n)}\underset{\sim}{u}^{(n)\prime} = \frac{1}{n}\underset{\sim}{I}_n$.

There is a close relationship between the chi-squared distribution and Dirichlet distribution from the above discussion. The following are further results.

Lemma 2. Assume that $(z_1,\ldots,z_m) \sim D_{m+1}(\frac{1}{2}\alpha_1,\ldots,\frac{1}{2}\alpha_m; \frac{1}{2}\alpha_{m+1})$ with $\alpha_i > 0$, $i = 1,\ldots,m+1$, and $\Sigma_1^{m+1}\alpha_i = n$ and $y_0,y_1,\ldots,y_{m+1}$ are distributed as chi-squared distributions with degrees of freedom $n,\alpha_1,\ldots,\alpha_{m+1}$, respectively; then

$$\phi_{\ell_n z_1,\ldots,\ell_n z_m}(t_1,\ldots,t_m) = \frac{\prod_1^m \phi_{\ell_n y_j}(t_j)}{\phi_{\ell_n y_0}(\sum_1^m t_j)} \tag{3.9}$$

$$= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+i(t_1+\cdots+t_m))}\prod_{j=1}^m \frac{\Gamma(\frac{\alpha_j}{2}+it_j)}{\Gamma(\frac{\alpha_j}{2})},$$

where $\phi(\cdot)$ denotes the characteristic function and $i = \sqrt{-1}$.

Proof. Let $\underset{\sim}{x} \sim N_n(\underset{\sim}{0},\underset{\sim}{I}_n)$ and $\underset{\sim}{x}_{(1)},\ldots,\underset{\sim}{x}_{(m+1)}$ have the similar sense as (3.2). Then $\|x_{(j)}\|^2 \overset{d}{=} y_j$, $j = 1,\ldots,m+1$, and $\|x\|^2 \overset{d}{=} y_0$. We have

$$\begin{pmatrix}\|\underset{\sim}{x}_{(1)}\|^2\\ \vdots\\ \|\underset{\sim}{x}_{(m)}\|^2\end{pmatrix} \overset{d}{=} \begin{pmatrix}\|\underset{\sim}{x}_{(1)}\|^2/\|\underset{\sim}{x}\|^2\\ \vdots\\ \|\underset{\sim}{x}_{(m)}\|^2/\|\underset{\sim}{x}\|^2\end{pmatrix} \cdot \|\underset{\sim}{x}\|^2 ,$$

and

$$\begin{pmatrix}\ell_n\|\underset{\sim}{x}_{(1)}\|^2\\ \vdots\\ \ell_n\|\underset{\sim}{x}_{(m)}\|^2\end{pmatrix} \overset{d}{=} \ell_n\|\underset{\sim}{x}\|^2\,\underset{\sim}{e}_n + \begin{pmatrix}\ell_n\|\underset{\sim}{x}_{(1)}\|^2/\|\underset{\sim}{x}\|^2\\ \vdots\\ \ell_n\|\underset{\sim}{x}_{(m)}\|^2/\|\underset{\sim}{x}\|^2\end{pmatrix} .$$

The first part of (3.9) follows from the fact that $\|\underset{\sim}{x}\|$ and $\underset{\sim}{x}/\|\underset{\sim}{x}\|$ are independent and $(\|\underset{\sim}{x}_{(1)}\|^2/\|\underset{\sim}{x}\|^2,\ldots,\|\underset{\sim}{x}_{(m)}\|^2/\|\underset{\sim}{x}\|^2) \overset{d}{=} (z_1,\ldots,z_m)$. Note (cf. Press (1969))

$$E(e^{it\ell_n y_j}) = \frac{2^{it}\Gamma(\frac{\alpha_j}{2}+it)}{\Gamma(\frac{\alpha_j}{2})} ;$$

the second part of of (3.9) follows. Q.E.D.

In particular, if $z \sim B(\frac{k}{2}, \frac{n-k}{2})$, $y_0 \sim \chi_n^2$ and $y_1 \sim \chi_k^2$, then

$$(3.10) \qquad \phi_{\ell_n z}(t) = \frac{\phi_{\ell_n y_1}(t)}{\phi_{\ell_n y_0}(t)} = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k}{2}+it)}{\Gamma(\frac{k}{2})\Gamma(\frac{n}{2}+it)} = \frac{B(\frac{k}{2}+it, \frac{n-k}{2})}{B(\frac{k}{2}, \frac{n-k}{2})} ,$$

where $B(a,b)$ is Beta function.

<u>Lemma 3</u>. Assume that $\underset{\sim}{x} \sim N_n(\underset{\sim}{0},\underset{\sim}{I}_n)$, $(z_1,\ldots,z_{n-1}) \sim D_n(\frac{1}{2},\ldots,\frac{1}{2};\frac{1}{2})$, $z_n = 1-\Sigma_1^{n-1} z_i$, $y_1,\ldots,y_n$ are i.i.d., $y_i \sim \chi_1^2$, $i = 1,\ldots,n$, and $\underset{\sim}{A}_1,\ldots,\underset{\sim}{A}_m$ are $n \times n$ symmetric matrices. Let $g_i(t_1,\ldots,t_n)$, $i = 1,\ldots,m$, be linear functions of $t_1,\ldots,t_n$, then

$$(3.11) \qquad \begin{pmatrix} \underset{\sim}{x}'\underset{\sim}{A}_1\underset{\sim}{x}/\|\underset{\sim}{x}\|^2 \\ \vdots \\ \underset{\sim}{x}'\underset{\sim}{A}_m\underset{\sim}{x}/\|\underset{\sim}{x}\|^2 \end{pmatrix} \overset{d}{=} \begin{pmatrix} g_1(z_1,\ldots,z_n) \\ \vdots \\ g_m(z_1,\ldots,z_n) \end{pmatrix} \Leftrightarrow \begin{pmatrix} \underset{\sim}{x}'\underset{\sim}{A}_1\underset{\sim}{x} \\ \vdots \\ \underset{\sim}{x}'\underset{\sim}{A}_m\underset{\sim}{x} \end{pmatrix} \overset{d}{=} \begin{pmatrix} g_1(y_1,\ldots,y_n) \\ \vdots \\ g_m(y_1,\ldots,y_n) \end{pmatrix} .$$

<u>Proof</u>. We firstly prove the implication towards the left. If the right side of (3.11) holds then

$$\|\underset{\sim}{x}\|^2 \begin{pmatrix} \underset{\sim}{x}'\underset{\sim}{A}_1\underset{\sim}{x}/\|\underset{\sim}{x}\|^2 \\ \vdots \\ \underset{\sim}{x}'\underset{\sim}{A}_m\underset{\sim}{x}/\|\underset{\sim}{x}\|^2 \end{pmatrix} = \begin{pmatrix} \underset{\sim}{x}'\underset{\sim}{A}_1\underset{\sim}{x} \\ \vdots \\ \underset{\sim}{x}'\underset{\sim}{A}_m\underset{\sim}{x} \end{pmatrix} \overset{d}{=} \begin{pmatrix} g_1(y_1,\dots,y_n) \\ \vdots \\ g_m(y_1,\dots,y_n) \end{pmatrix} \overset{d}{=} \begin{pmatrix} g_1(x_1^2,\dots,x_n^2) \\ \vdots \\ g_m(x_1^2,\dots,x_n^2) \end{pmatrix} = \|\underset{\sim}{x}\|^2 \begin{pmatrix} g_1(\frac{x_1^2}{\|\underset{\sim}{x}\|^2},\dots,\frac{x_n^2}{\|\underset{\sim}{x}\|^2}) \\ \vdots \\ g_m(\frac{x_1^2}{\|\underset{\sim}{x}\|^2},\dots,\frac{x_n^2}{\|\underset{\sim}{x}\|^2}) \end{pmatrix}$$

and from Lemma 1

$$\begin{pmatrix} \underset{\sim}{x}'\underset{\sim}{A}_1\underset{\sim}{x}/\|\underset{\sim}{x}\|^2 \\ \vdots \\ \underset{\sim}{x}'\underset{\sim}{A}_m\underset{\sim}{x}/\|\underset{\sim}{x}\|^2 \end{pmatrix} \overset{d}{=} \begin{pmatrix} g_1(\frac{x_1^2}{\|\underset{\sim}{x}\|^2},\dots,\frac{x_n^2}{\|\underset{\sim}{x}\|^2}) \\ \vdots \\ g_m(\frac{x_1^2}{\|\underset{\sim}{x}\|^2},\dots,\frac{x_n^2}{\|\underset{\sim}{x}\|^2}) \end{pmatrix} \overset{d}{=} \begin{pmatrix} g_1(z_1,\dots,z_n) \\ \vdots \\ g_m(z_1,\dots,z_n) \end{pmatrix} .$$

Here we use the fact again that $\|\underset{\sim}{x}\|^2$ and $\underset{\sim}{x}/\|\underset{\sim}{x}\|$ are independent. The proof in the other direction is similar. Q.E.D.

By Lemma 3, we can change the theory of distributions of quadratic forms from chi-square distributions to Dirichlet distributions.

<u>Corollary 1</u>. Assume that $\underset{\sim}{x} \sim N_n(\underset{\sim}{0},\underset{\sim}{I})$ and $\underset{\sim}{A}$ is an $n \times n$ symmetric matrix, then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}/\|\underset{\sim}{x}\|^2 \sim B(\frac{k}{2}, \frac{n-k}{2})$ if and only if $\underset{\sim}{A}^2 = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$.

<u>Proof</u>. By Cochran's Theorem, $\underset{\sim}{A}^2 = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$ if and only if $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim \chi_k^2$ or $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \overset{d}{=} \Sigma_1^k y_i$, hence if and only if $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}/\|\underset{\sim}{x}\|^2 = \Sigma_1^k z_i \sim B(\frac{k}{2}, \frac{n-k}{2})$ from Lemma 3 with $m = 1$ and $g_1(t_1,\dots,t_n) = t_1 + \dots + t_k$. Q.E.D.

Corollary 2. Assume that $\underset{\sim}{x} \sim N_n(\underset{\sim}{0}, \underset{\sim}{I}_n)$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are $n \times n$ symmetric matrices, then $z_1, \ldots, z_{n-1}$ has the Dirichlet distribution $D_n(\frac{1}{2}, \ldots, \frac{1}{2}; \frac{1}{2})$ and $\Sigma_1^n z_i = 1$, then

$$(3.12) \qquad \left(\frac{\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}}{\|\underset{\sim}{x}\|^2}, \frac{\underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}}{\|\underset{\sim}{x}\|^2}\right) \stackrel{d}{=} \left(\sum_1^k \lambda_i z_i, \sum_{k+1}^{k+m} \nu_j z_j\right), \quad k+m \le n,$$

if and only if $\underset{\sim}{A}\underset{\sim}{B} = 0$, where λ_i and ν_j are nonzero real numbers.

Proof. By Craig's Theorem (cf. Anderson and Styan (1980)) $\underset{\sim}{A}\underset{\sim}{B} = 0$ if and only if $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ and $\underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}$ are independent or if and only if

$$(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}, \underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}) \stackrel{d}{=} \left(\sum_1^k \lambda_i y_i, \sum_1^m \nu_j y_{j+k}\right).$$

where $\lambda_1, \ldots, \lambda_k$ and $\nu_1, \ldots, \nu_m$ are the nonzero eigenvalues of $\underset{\sim}{A}$ and $\underset{\sim}{B}$, respectively. Thus the corollary follows by Lemma 3. Q.E.D.

Corollary 3. Let $\underset{\sim}{x}$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be defined as in Corollary 2, then $(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}/\|\underset{\sim}{x}\|^2, \underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}/\|\underset{\sim}{x}\|^2) \sim D_3(\frac{k}{2}, \frac{m}{2}; \frac{n-k-m}{2})$, with $k > 0$, $m > 0$, $k+m \le n$, if and only if $\underset{\sim}{A}\,\underset{\sim}{B} = 0$, $\underset{\sim}{A}^2 = \underset{\sim}{A}$, $\underset{\sim}{B}^2 = \underset{\sim}{B}$, $rk(\underset{\sim}{A}) = k$ and $rk(\underset{\sim}{B}) = m$.

Proof. The corollary follows from Corollary 2 and $(\Sigma_1^k z_i, \Sigma_{k+1}^{k+m} z_i) \sim D_3(\frac{k}{2}, \frac{m}{2}; \frac{n-k-m}{2})$. Q.E.D.

4. Distributions of Quadratic Forms and Cochran's Theorem.

In this section we want to extend distributions of quadratic forms and Cochran's Theorem to the case of elliptically contoured distributions with mean zero. Firstly, we need to generalize the Dirichlet distribution and the Beta distribution.

If a random vector $(z_1,\ldots,z_m)'$ satisfies $(z_1,\ldots,z_m) \overset{d}{=} R^2(u_1,\ldots,u_m)$, where $R \sim F(x)$, R is independent of $(u_1,\ldots,u_{m-1})$, $\Sigma_1^m u_i = 1$, and $(u_1,\ldots,u_{m-1}) \sim D_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m)$, then we write $(z_1,\ldots,z_{m-1}) \sim G_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m;\phi)$ and $(z_1,\ldots,z_m) \sim G_m(\alpha_1,\ldots,\alpha_{m-1},\alpha_m;\phi)$, where ϕ is related to $F(x)$ as in (1.1) with $n = 2(\alpha_1 + \cdots + \alpha_m)$.

It is easy to show that the density of $G_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m;\phi)$ is

$$(4.1) \qquad \frac{\Gamma(\frac{n}{2})}{\prod_1^m \Gamma(\alpha_i)} \prod_1^{m-1} z_i^{\alpha_i - 1} \int_{\sqrt{\sum_1^{m-1} z_i}}^{\infty} r^{-(n-2)} \left(r^2 - \sum_1^{m-1} z_i \right)^{\alpha_m - 1} dF(r) \text{ , if } z_i \geq 0 \text{ .}$$

Further, if R has a density $f(r)$, then the joint density of $u_1,\ldots,u_{m-1}$ and R is (cf. (3.1))

$$\frac{\Gamma(\frac{n}{2})}{\prod_{i=1}^m \Gamma(\alpha_i)} \prod_{i=1}^{m-1} u_i^{\alpha_i - 1} \left(1 - \sum_1^{m-1} u_i \right)^{\alpha_m - 1} f(r) \text{ .}$$

Consider the following transformation:

$$z_i = r^2 u_i, \; i = 1,\ldots,m-1 \text{ ,}$$
$$z_m = r^2 \left(1 - \sum_1^{m-1} u_i \right),$$

then

$$u_i = z_i / r^2, \; i = 1,\ldots,m-1 \text{ ,}$$
$$r = \left(\sum_1^m z_i \right)^{1/2} \text{ .}$$

Thus the Jacobian of the transformation is

$$\operatorname{mod}\begin{vmatrix} r^{\frac12} & & & \\ & r^{\frac12} & & \\ & & \ddots & r^{\frac12} \\ \frac{1}{2}\left(\sum_1^m z_i\right)^{-\frac12} & \cdots & & \frac{1}{2}\left(\sum_1^m z_i\right)^{-\frac12} \end{vmatrix} = r^{-2(m-1)}\ \frac{1}{2}\left(\sum_1^m z_i\right)^{-\frac12} = \frac{1}{2}\left(\sum_1^m z_i\right)^{-m+\frac12} .$$

Now the joint density of $z_1,\ldots,z_m$ is

$$\text{(4.2)}\qquad \frac{\Gamma(\frac{n}{2})}{2\prod_1^m \Gamma(\alpha_i)} \prod_{i=1}^m z_i^{\alpha_i-1} \left(\sum_1^m z_i\right)^{-(n-1)/2} f\left[\left(\sum_1^m z_i\right)^{\frac12}\right] .$$

If $\underset{\sim}{x} \sim EC_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma},\underset{\sim}{\phi})$, then R has a density $f(r)$ if and only if $\underset{\sim}{x}$ has a density which must have the form

$$|\underset{\sim}{\Sigma}|^{-\frac12} g((\underset{\sim}{x}-\underset{\sim}{\mu})'\ \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}))$$

for a suitable function $g(\cdot)$. Moreover there exists a relationship between $f(\cdot)$ and $g(\cdot)$, i.e.,

$$f(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1} g(r^2) .$$

(cf. Cambanis, Huang and Simons (1981)). Substituting the above formula into (4.2), the density of $z_1,\ldots,z_m$ becomes

$$(4.3) \qquad \frac{\pi^{n/2}}{\prod_1^m \Gamma(\alpha_i)} \prod_{i=1}^m z_i^{\alpha_i - 1} \, g\left(\sum_1^m z_i\right) .$$

In Section 6 we will give some examples of $G_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m;\phi)$. Anderson and Fang (1982) give some further applications of $G_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m;\phi)$.

<u>Theorem 1</u>. Suppose that $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\phi)$, $p(\underset{\sim}{x}=\underset{\sim}{0}) = 0$ and $\underset{\sim}{A}$ is an $n\times n$ symmetric matrix, then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim G_2(\frac{k}{2};\frac{n-k}{2};\phi)$ if and only if $\underset{\sim}{A}^2 = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$.

<u>Proof</u>. If $\underset{\sim}{A}^2 = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$, then there exists an orthogonal matrix $\underset{\sim}{\Gamma}$ such that $\underset{\sim}{\Gamma}'\underset{\sim}{A}\underset{\sim}{\Gamma} = \begin{pmatrix} \underset{\sim}{I}_k & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix}$. Let $\underset{\sim}{y} = \underset{\sim}{\Gamma}'\underset{\sim}{x}$; it can be verified that that $\underset{\sim}{y} \sim EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\phi)$ and $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} = \Sigma_1^k y_i^2$, where $\underset{\sim}{y} = (y_1,\ldots,y_n)'$. Thus

$$(4.4) \qquad \underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} = \sum_1^k y_i^2 = \|\underset{\sim}{y}\|^2 \cdot \frac{\sum_1^k y_i^2}{\|\underset{\sim}{y}\|^2} .$$

As is known, $\|\underset{\sim}{y}\|^2$ and $\Sigma_1^k y_i^2/\|\underset{\sim}{y}\|^2$ are independent and the distribution of $\Sigma_1^k y_i^2/\|\underset{\sim}{y}\|^2$ does not depend on what the distribution of $\underset{\sim}{y}$ is in the class of $EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\phi)$; therefore we can assume $\underset{\sim}{y} \sim N_n(\underset{\sim}{0},\underset{\sim}{I}_n)$. It implies that $\Sigma_1^k y_i^2/\|\underset{\sim}{y}\|^2 \sim B(\frac{k}{2},\frac{n-k}{2})$. As $\|\underset{\sim}{y}\| \overset{d}{=} R \sim F(x)$, we have completed the proof of the "if" part.

If $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim G_2(\frac{k}{2};\frac{n-k}{2};\phi)$, write

$$\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} = \|\underset{\sim}{x}\|^2 \cdot \frac{\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}}{\|\underset{\sim}{x}\|^2} \overset{d}{=} R^2 \cdot z ,$$

where $R \sim F(x)$, $z \sim B(\frac{k}{2},\frac{n-k}{2})$ and they are independent. By Lemma 1 we have

$$\frac{\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}}{\|\underset{\sim}{x}\|^2} \stackrel{d}{=} z .$$

From Corollary 1 of Lemma 3 the assertion follows. Q.E.D.

Corollary. Assume that $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$ and $p(\underset{\sim}{x}=\underset{\sim}{0}) = 0$, and $\underset{\sim}{A}$ is an $n \times n$ symmetric matrix; then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim G_2(\frac{k}{2}; \frac{n-k}{2}; \phi)$ if and only if $\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A} = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$.

Proof. As $\underset{\sim}{\Sigma} > 0$, there exists $\underset{\sim}{\Sigma}^{\frac{1}{2}} > 0$ such that $\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{\Sigma}^{\frac{1}{2}} = \underset{\sim}{\Sigma}$. Let $\underset{\sim}{y} = \underset{\sim}{\Sigma}^{-\frac{1}{2}}\underset{\sim}{x}$, then $\underset{\sim}{y} \sim EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\phi)$. By Theorem 1

$$\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{y}'(\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}})\underset{\sim}{y} \sim G_2(\frac{k}{2}; \frac{n-k}{2}; \phi)$$

if and only if $(\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}})^2 = \underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}}$ and $rk(\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}}) = k$, i.e. $\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A} = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$. Q.E.D.

It is well-known that $\underset{\sim}{A}^2 = \underset{\sim}{A}$ if and only if $rk(\underset{\sim}{A}) + rk(\underset{\sim}{I}-\underset{\sim}{A}) = n$. Hence $\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A} = \underset{\sim}{A}$ if and only if $rk(\underset{\sim}{A}) + rk(\underset{\sim}{\Sigma}-\underset{\sim}{A}) = n$. The corollary shows us that $\underset{\sim}{\Sigma}$ must be a generalized inverse of $\underset{\sim}{A}$. Kelker (1970) gives a result similar to the corollary, but he assumes finite fourth moments for the components of $\underset{\sim}{x}$.

If $\underset{\sim}{x} \sim N_n(\underset{\sim}{0},\underset{\sim}{I}_n)$, $\underset{\sim}{A}' = \underset{\sim}{A}$, $\underset{\sim}{A}^2 = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$, then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim \chi_k^2$ by Cochran's Theorem, i.e. $G_2(\frac{k}{2}; \frac{n-k}{2}; \phi) = \chi_k^2$ if $\phi(t) = \exp(-t/2)$. The inverse proposition also holds.

Theorem 2. The distribution $G_2(\frac{k}{2}, \frac{n-k}{2}; \phi)$ is χ_k^2 if and only if $\phi(t) = \exp(-t/2)$.

Proof. We only prove the "only if" part. If $z \sim G_2(\frac{k}{2}; \frac{n-k}{2}; \phi)$ is χ_k^2, by the definition of $G_2(\frac{k}{2}; \frac{n-k}{2}; \phi)$. We have $z \overset{d}{=} R^2 \cdot w$, where $R \sim F(x)$, $w \sim B(\frac{k}{2}, \frac{n-k}{2})$ and R^2 is independent of w. Let $y_1 \sim \chi_k^2$ and $y_0 \sim \chi_n^2$; from (3.10) and the supposition we find

$$\phi_{\ell_n y_1}(t) = \phi_{\ell_n z}(t) = \phi_{\ell_n R^2}(t) \cdot \phi_{\ell_n w}(t) = \phi_{\ell_n R^2}(t) \frac{\phi_{\ell_n y_1}(t)}{\phi_{\ell_n y_0}(t)} ,$$

which implies $R^2 \sim \chi_n^2$, i.e. $\phi(t) = \exp(-t/2)$. Q.E.D.

By a method similar to those used in proving Theorem 1 and Theorem 2, we obtain the following theorems and corollaries.

Theorem 3. Suppose that $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0}, \underset{\sim}{I}_n, \phi)$, $p(\underset{\sim}{x} = \underset{\sim}{0}) = 0$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are $n \times n$ symmetric matrices, then $(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}, \underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}) \sim G_3(\frac{k}{2}, \frac{\ell}{2}; \frac{n-k-\ell}{2}; \phi)$ if and only if $\underset{\sim}{A}\underset{\sim}{B} = \underset{\sim}{0}$, $\underset{\sim}{A}^2 = \underset{\sim}{A}$, $\underset{\sim}{B}^2 = \underset{\sim}{B}$, $rk(\underset{\sim}{A}) = k$ and $rk(\underset{\sim}{B}) = \ell$.

Corollary 1. Suppose that $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0}, \underset{\sim}{\Sigma}, \phi)$ with $\underset{\sim}{\Sigma} > 0$, $p(\underset{\sim}{x} = \underset{\sim}{0}) = 0$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are $n \times n$ symmetric matrices, then $(\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}, \underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}) \sim G_3(\frac{k}{2}, \frac{\ell}{2}; \frac{n-k-\ell}{2}; \phi)$ if and only if $\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A} = \underset{\sim}{A}$, $\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B} = \underset{\sim}{B}$, $rk(\underset{\sim}{A}) = k$, $rk(\underset{\sim}{B}) = \ell$ and $\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{B} = \underset{\sim}{0}$.

Corollary 2. Assume that $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0}, \underset{\sim}{I}_n, \phi)$, $p(\underset{\sim}{x} = \underset{\sim}{0}) = 0$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are projection matrices with $rk(\underset{\sim}{A}) = k$ and $rk(\underset{\sim}{B}) = \ell$ satisfying $\underset{\sim}{A}\underset{\sim}{B} = \underset{\sim}{0}$, then

$$\frac{\ell}{k} \frac{\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}}{\underset{\sim}{x}'\underset{\sim}{B}\underset{\sim}{x}} \sim F(k, \ell) .$$

Proof. As $\underset{\sim}{A}\underset{\sim}{B} = \underset{\sim}{0}$, there exists an orthogonal matrix $\underset{\sim}{\Gamma}$ such that $\underset{\sim}{\Gamma}'\underset{\sim}{A}\underset{\sim}{\Gamma} = \mathrm{diag}(\underbrace{1,\ldots,1}_{k},0,\ldots,0)$ and $\underset{\sim}{\Gamma}'\underset{\sim}{B}\underset{\sim}{\Gamma} = \mathrm{diag}(\underbrace{0,\ldots,0}_{k},\underbrace{1,\ldots,1}_{\ell},0,\ldots,0)$.

Let $\underset{\sim}{y} = \underset{\sim\sim}{\Gamma x}$, then $\underset{\sim}{y} \sim EC_n(\underset{\sim}{0}, \underset{\sim}{I}_n, \phi)$, $\underset{\sim}{x}'\underset{\sim\sim}{Ax} = \Sigma_1^k y_i^2$ and $\underset{\sim}{x}'\underset{\sim\sim}{Bx} = \Sigma_{k+1}^{k+\ell} y_i^2$, thus

$$\frac{\ell}{k}\frac{\underset{\sim}{x}'\underset{\sim\sim}{Ax}}{\underset{\sim}{x}'\underset{\sim\sim}{Bx}} = \frac{\ell}{k}\frac{\sum_1^k y_i^2}{\sum_{k+1}^{k+\ell} y_i^2} = \frac{\ell}{k}\frac{\sum_1^k y_i^2/\|\underset{\sim}{y}\|^2}{\sum_{k+1}^{k+\ell} y_i^2/\|\underset{\sim}{y}\|^2} \overset{d}{=} \frac{\ell}{k}\frac{\sum_1^k z_i^2/\|\underset{\sim}{z}\|^2}{\sum_{k+1}^{k+\ell} z_i^2/\|\underset{\sim}{z}\|^2} = \frac{\ell}{k}\frac{\sum_1^k z_i^2}{\sum_{k+1}^{k+\ell} z_i^2} \sim F(k,\ell) ,$$

where $\underset{\sim}{z} \sim N_n(\underset{\sim}{0}, \underset{\sim}{I}_n)$. Q.E.D.

Kelker (1970) treated the distribution of $\Sigma_1^k \chi_i^2/\Sigma_{k+1}^{k+\ell} \chi_i^2$ by a different method.

<u>Theorem 4</u>. The joint distribution $G_3(\frac{k}{2}, \frac{\ell}{2}; \frac{n-k-\ell}{2}; \phi)$ is the product of the distributions of χ_k^2 and χ_ℓ^2 if and only if $\phi(t) = \exp(-t/2)$.

<u>Theorem 5</u>. Suppose that $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0}, \underset{\sim}{I}_n, \phi)$, $P(\underset{\sim}{x}=\underset{\sim}{0}) = 0$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are $n \times n$ symmetric matrices and $(z_1, \ldots, z_{n-1}) \sim D_n(\frac{1}{2}, \ldots, \frac{1}{2}; \frac{1}{2})$ is independent of R. Then

$$\begin{pmatrix} \underset{\sim}{x}'\underset{\sim\sim}{Ax} \\ \underset{\sim}{x}'\underset{\sim\sim}{Bx} \end{pmatrix} \overset{d}{=} \begin{pmatrix} \sum_1^k \lambda_i z_i \\ \sum_{k+1}^{k+\ell} \nu_j z_j \end{pmatrix}$$

with real numbers λ_i and ν_j if and only if $\underset{\sim\sim}{AB} = \underset{\sim}{0}$ and λ_i's are the nonzero eigenvalues of $\underset{\sim}{A}$, ν_j's are the nonzero eigenvalues of $\underset{\sim}{B}$.

The above theorems can be generalized to the case of several quadratic forms.

5. Tripotent Matrices.

A square matrix $\underset{\sim}{A}$ is said to be tripotent whenever $\underset{\sim}{A}^3 = \underset{\sim}{A}$. Anderson and Styan (1980) extended Cochran's Theorem to tripotent matrices. If $\underset{\sim}{A}^3 = \underset{\sim}{A}$, the eigenvalues of $\underset{\sim}{A}$ are 1, -1 and 0. Let p and q denote the number of eigenvalues equal to 1 and -1, respectively. If $\underset{\sim}{x} \sim N_n(\underset{\sim}{0}, \underset{\sim}{I}_n)$, then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \overset{d}{=} y_1 - y_2$, where y_1 and y_2 are independent, $y_1 \sim \chi^2_p$ and $y_2 \sim \chi^2_q$. The distribution of $y_1 - y_2$ has been given by Press (1969) and Robinson (1965) and we denote it by $H(\frac{p}{2}, \frac{q}{2})$. Similarly, if $(v_1, v_2) \sim D_3(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$ we denote the distribution of $z = v_1 - v_2$ by $H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$.

Lemma 4. Assume that $\underset{\sim}{x} \sim N_n(\underset{\sim}{0}, \underset{\sim}{I}_n)$ and $\underset{\sim}{A}$ is an $n \times n$ symmetric matrix. Then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$ if and only if $\underset{\sim}{A}^3 = \underset{\sim}{A}$ with p 1's and q -1's as its nonzero eigenvalues.

Proof. Apply Lemma 3 with $g_1(t_1, \ldots, t_n) = \Sigma_1^p t_i - \Sigma_{p+1}^{p+q} t_j$ and $m = 1$ which completes the proof.

Now we generalize the distribution $H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$ to the case of elliptically contoured distributions. If $z \overset{d}{=} v_1 - v_2 \sim H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2})$, $R \sim F(x)$ and is independent of z, we denote the distribution of $R^2 z$ by $H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2}; \phi)$ where ϕ is related to F(x) as in (1.1). By the same technique used in the proof of Theorem 1 and Lemma 4 we obtain the following theorem.

Theorem 6. Suppose that $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0}, \underset{\sim}{I}_n, \phi)$, $p(\underset{\sim}{x} = \underset{\sim}{0}) = 0$ and $\underset{\sim}{A}$ is an $n \times n$ symmetric matrix, then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2}; \phi)$ if and only if $\underset{\sim}{A}^3 = \underset{\sim}{A}$ with p 1's and q -1's as its eigenvalues.

Corollary. Suppose that $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0},\underset{\sim}{\Sigma},\phi)$ with $\underset{\sim}{\Sigma} > 0$, $p(\underset{\sim}{x}=\underset{\sim}{0}) = 0$, and $\underset{\sim}{A}$ is an $n \times n$ symmetric matrix, then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2}; \phi)$ if and only if

$$(5.1) \qquad rk(\underset{\sim}{A}) = rk(\underset{\sim}{A}-\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A}) + rk(\underset{\sim}{A}+\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A})$$

and $\underset{\sim}{A}\underset{\sim}{\Sigma}$ has p 1's and q -1's as its eigenvalues.

Proof. As $\underset{\sim}{\Sigma} > 0$, there exists $\underset{\sim}{\Sigma}^{\frac{1}{2}} > 0$ such that $\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{\Sigma}^{\frac{1}{2}} = \underset{\sim}{\Sigma}$. Let $\underset{\sim}{x} = \underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{y}$, then $\underset{\sim}{y} \sim EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\phi)$ and $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} = \underset{\sim}{y}'(\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}})\underset{\sim}{y}$. From Theorem 6 $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim H(\frac{p}{2}, \frac{q}{2}; \frac{n-p-q}{2}; \phi)$ if and only if

(i) $\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}}$ has p 1's and q -1's as its eigenvalues, or equivalently $\underset{\sim}{A}\underset{\sim}{\Sigma}$ has p 1's and q -1's as its eigenvalues.

(ii) $(\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}})^3 = (\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}})$. Anderson and Styan (1980) point out that $\underset{\sim}{B}^3 = \underset{\sim}{B}$ if and only if $rk(\underset{\sim}{B}) = rk(\underset{\sim}{B}+\underset{\sim}{B}^2) + rk(\underset{\sim}{B}-\underset{\sim}{B}^2)$ for any square matrix $\underset{\sim}{B}$. Thus $(\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}})^3 = \underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}}$ if and only if

$$\begin{aligned} rk(\underset{\sim}{A}) &= rk(\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}}) = rk(\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}}+\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}}) + rk(\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}}-\underset{\sim}{\Sigma}^{\frac{1}{2}}\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A}\underset{\sim}{\Sigma}^{\frac{1}{2}}) \\ &= rk(\underset{\sim}{A}+\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A}) + rk(\underset{\sim}{A}-\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A}) \ . \qquad \text{Q.E.D.} \end{aligned}$$

6. Applications.

In this section some applications are given. In the first part we consider the theory on linear regression with the error being distributed according to an elliptically contoured distribution. We will find the distribution of the sample variance, the distribution of the ratio of the sample mean to the sample standard deviation in the second part. In addition we will give the distributions of quadratic forms for the multivariate t-distribution, the multivariate uniform distribution, etc.

6.1. Linear Regression Analysis. We consider the following model:

$$(6.1) \qquad \begin{cases} \underset{\sim}{y}_{n\times 1} = \underset{\sim}{X}_{n\times p}\underset{\sim}{\beta}_{p\times 1} + \underset{\sim}{e}_{n\times 1}, & rk(\underset{\sim}{X}) = p < n \\ E(\underset{\sim}{e}) = \underset{\sim}{0}, \quad E(\underset{\sim}{e}\underset{\sim}{e}') = \underset{\sim}{\Sigma} > 0 \\ \underset{\sim}{e} \sim EC_n(\underset{\sim}{0},\underset{\sim}{\Sigma},\underset{\sim}{\phi}) . \end{cases}$$

There exists a $p\times p$ matrix $\underset{\sim}{A}$ such that $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$ and

$$(6.2) \qquad \underset{\sim}{e} \overset{d}{=} R\underset{\sim}{A}'\underset{\sim}{u}^{(n)} .$$

Hence

$$(6.3) \qquad \underset{\sim}{y} \overset{d}{=} \underset{\sim}{X}\underset{\sim}{\beta} + R\underset{\sim}{A}'\underset{\sim}{u}^{(n)} ,$$

i.e., $\underset{\sim}{y} \sim EC_n(\underset{\sim}{X}\underset{\sim}{\beta},\underset{\sim}{\Sigma},\phi)$.

Minimizing $\underset{\sim}{e}'\underset{\sim}{e} = (\underset{\sim}{y}-\underset{\sim}{X}\underset{\sim}{\beta})'(\underset{\sim}{y}-\underset{\sim}{X}\underset{\sim}{\beta})$ with respect to $\underset{\sim}{\beta}$ gives the least squares estimator

$$(6.4) \qquad \hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}'\underset{\sim}{X})\underset{\sim}{X}'\underset{\sim}{y} .$$

In order to find the distribution of $\underset{\sim}{\beta}$ we need the following lemma.

Lemma 5. Assume $\underset{\sim}{y} \sim EC_n(\underset{\sim}{\mu},\underset{\sim}{I}_n,\underset{\sim}{\phi})$ and $\underset{\sim}{A}$ is $n \times p$ matrix with $rk(\underset{\sim}{A}) = p < n$; then $\underset{\sim}{X} \equiv \underset{\sim}{A}'\underset{\sim}{y} \sim EC_p(\underset{\sim}{A}'\underset{\sim}{\mu},\underset{\sim}{A}'\underset{\sim}{A},\phi)$, where $\phi \in \Phi_p$ corresponds to R^* and

$$R^* \overset{d}{=} R \cdot b , \tag{6.5}$$

where $b \geq 0$, $b^2 \sim B(\frac{1}{2}p , \frac{1}{2}(n-p))$, and b is independent of R.

Proof. Consider the singular value decomposition of $\underset{\sim}{A}$, i.e., there exist orthogonal matrices $\underset{\sim}{\Gamma}_{p\times p}$ and $\underset{\sim}{\Delta}_{n\times n}$ such that

$$\underset{\sim}{A}' = \underset{\sim}{\Gamma}(\underset{\sim}{D}_r \ \underset{\sim}{0})\underset{\sim}{\Delta}' ,$$

where $\underset{\sim}{D}_r = \text{diag}(\lambda_1,\ldots,\lambda_p)$ and $\lambda_1,\ldots,\lambda_p$ are the roots of $|\underset{\sim}{A}'\underset{\sim}{A}-\lambda\underset{\sim}{I}| = 0$. From the assumption

$$\underset{\sim}{y} \overset{d}{=} \underset{\sim}{\mu} + R\underset{\sim}{u}^{(n)} ,$$

thus

$$\begin{aligned}
\underset{\sim}{x} &\overset{d}{=} \underset{\sim}{A}'\underset{\sim}{\mu} + R\underset{\sim}{A}'\underset{\sim}{u}^{(n)} \\
&= \underset{\sim}{A}'\underset{\sim}{\mu} + R\underset{\sim}{\Gamma}(\underset{\sim}{D}_\lambda \ \underset{\sim}{0})\underset{\sim}{\Delta}'\underset{\sim}{u}^{(n)} \\
&\overset{d}{=} \underset{\sim}{A}'\underset{\sim}{\mu} + R\underset{\sim}{\Gamma}(\underset{\sim}{D}_\lambda \ \underset{\sim}{0})\underset{\sim}{u}^{(n)} ,
\end{aligned}$$

because $\underset{\sim}{u}^{(n)} \overset{d}{=} \underset{\sim}{\Delta}'\underset{\sim}{u}^{(n)}$. Let $\underset{\sim}{z} \sim N_n(\underset{\sim}{0},\underset{\sim}{I}_n)$ and

$$\underset{\sim}{u}^{(n)} = \begin{pmatrix} \underset{\sim}{u}_{(1)} \\ \underset{\sim}{u}_{(2)} \end{pmatrix} , \quad \underset{\sim}{z} = \begin{pmatrix} \underset{\sim}{z}_{(1)} \\ \underset{\sim}{z}_{(2)} \end{pmatrix} ,$$

where $\underset{\sim}{u}_{(1)}$ and $\underset{\sim}{z}_{(1)}$ are $p \times 1$ vectors. Based on the relationship between $\underset{\sim}{u}^{(n)}$ and $\underset{\sim}{z}$ we have

$$\begin{aligned}
\underset{\sim}{x} &\overset{d}{=} \underset{\sim}{A}'\underset{\sim}{\mu} + R\underset{\sim}{\Gamma}\underset{\sim}{D}_\lambda\underset{\sim}{u}_{(1)} \overset{d}{=} \underset{\sim}{A}'\underset{\sim}{\mu} + R\underset{\sim}{\Gamma}\underset{\sim}{D}_\lambda\underset{\sim}{z}_{(1)}/\|\underset{\sim}{z}\| \\
&= \underset{\sim}{A}'\underset{\sim}{\mu} + R\underset{\sim}{\Gamma}\underset{\sim}{D}_\lambda(\|\underset{\sim}{z}_{(1)}\|/\|\underset{\sim}{z}\|)\underset{\sim}{z}_{(1)}/\|\underset{\sim}{z}_{(1)}\| \\
&\overset{d}{=} \underset{\sim}{A}'\underset{\sim}{\mu} + (Rb)(\underset{\sim}{\Gamma}\underset{\sim}{D}_\lambda)\underset{\sim}{u}^{(p)} .
\end{aligned}$$

Since $\underset{\sim}{A}'\underset{\sim}{A} = (\underset{\sim}{\Gamma}\underset{\sim}{D}_\lambda)(\underset{\sim}{\Gamma}\underset{\sim}{D}_\lambda)'$, the Lemma follows. Q.E.D.

By Lemma 5 and (6.4) we have

$$\hat{\underset{\sim}{\beta}} \overset{d}{=} (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'[\underset{\sim}{X}\underset{\sim}{\beta} + R\underset{\sim}{A}'\underset{\sim}{u}^{(n)}] \tag{6.6}$$

$$= \underset{\sim}{\beta} + R(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{A}'\underset{\sim}{u}^{(n)} ,$$

or

$$\hat{\underset{\sim}{\beta}} \sim EC_p(\underset{\sim}{\beta}, (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{\Sigma}\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}, \phi) ,$$

where ϕ is defined by Lemma 5. In particular, if $\underset{\sim}{\Sigma} = \sigma^2\underset{\sim}{I}_p$, then

$$\hat{\underset{\sim}{\beta}} \sim EC_p(\underset{\sim}{\beta}, \sigma^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1}, \phi) .$$

Denote

$$s = (\underset{\sim}{y} - \underset{\sim}{X}\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - \underset{\sim}{X}\hat{\underset{\sim}{\beta}}) \tag{6.7}$$

$$= \underset{\sim}{y}'[\underset{\sim}{I} - \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}']\underset{\sim}{y}$$

$$= \underset{\sim}{e}'[\underset{\sim}{I} - \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}']\underset{\sim}{e} .$$

If $\underset{\sim}{e} \sim EC_n(\underset{\sim}{0}, \sigma^2\underset{\sim}{I}_n, \phi)$, $p(\underset{\sim}{e}=\underset{\sim}{0}) = 0$ and $rk\underset{\sim}{X} = p$, then $s \sim \sigma^2 G_2(\frac{n-p}{2}; \frac{p}{2}; \phi)$ by Theorem 1. We summarize the above results in the following theorem.

<u>Theorem 7</u>. Assume $\underset{\sim}{e} \sim EC_n(\underset{\sim}{0}, \sigma^2\underset{\sim}{I}_n, \phi)$, $p(\underset{\sim}{e}=\underset{\sim}{0}) = 0$ and $rk\underset{\sim}{X} = p < n$, then

$$\hat{\underset{\sim}{\beta}} \sim EC_p(\underset{\sim}{\beta}, \sigma^2(\underset{\sim}{X}'\underset{\sim}{X})^{-1}, \phi)$$

and

$$s \sim \sigma^2 G_2(\tfrac{n-p}{2}; \tfrac{p}{2}; \phi) .$$

Now we consider the linear hypothesis

$$\text{(6.8)} \qquad H\colon \; H\beta = c \; ,$$

where

$$H\colon\; q \times p, \; rk(H) = q < p \; .$$

Under the condition of $H\beta = c$ minimizing $e'e$ with respect to β gives the least squares estimator

$$\text{(6.9)} \qquad \beta_H = \hat{\beta} - (X'X)^{-1}H'(H(X'X)^{-1}(H\beta-c) \; ,$$

where β is given by (6.4), and the corresponding $(y-X\hat{\beta}_H)'(y-X\hat{\beta}_H)$ becomes

$$\text{(6.10)} \qquad s_H = (y-X\hat{\beta}_H)'(y-X\hat{\beta}_H) = (y-X\hat{\beta}+X\hat{\beta}-X\hat{\beta}_H)'(y-X\hat{\beta}+X\hat{\beta}-X\hat{\beta}_H)$$

$$= s + (H\hat{\beta}-c)'(H(X'X)^{-1}H')^{-1}(H\hat{\beta}-c) \; ,$$

where s is defined by (6.7).

The statistics testing the hypothesis (6.8) is

$$\text{(6.11)} \qquad \lambda = \frac{s_H - s}{s} \; .$$

When the hypothesis is true we have $c = H\beta$ and

$$\text{(6.12)} \qquad (H\hat{\beta}-c) = H(\hat{\beta}-\beta) \overset{d}{=} RH(X'X)^{-1}X'A'u^{(n)} \overset{d}{=} H(X'X)^{-1}X'e \; .$$

Thus

$$\text{(6.13)} \qquad s_H - s = e'[X(X'X)^{-1}H'(H(X'X)^{-1}H')^{-1}H(X'X)^{-1}X']e$$

$$\equiv e'D_H e \quad \text{(say)} \; .$$

It is easy to see that $D_H^2 = D_H$, $\mathrm{rk} D_H = q$, and $D_H(I-X(X'X)^{-1}X') = 0$. From (6.13) and (6.7) λ can be expressed as

$$(6.14) \qquad \lambda = \frac{e'D_H e}{e'[I-X(X'X)^{-1}X']e} ,$$

and

$$\frac{n-p}{q}\lambda \sim F(q,n-p) .$$

(cf. Corollary 2 of Theorem 3).

<u>Theorem 8</u>. Assume $e \sim EC_n(0,\sigma^2 I_n,\phi)$, $p(e=0) = 0$ and $\mathrm{rk} X = p$, when the hypothesis (6.8) is true the distribution of $(n-p)\lambda/q$ is $F(q,n-q)$.

<u>6.2. Some examples.</u>

<u>Example 1</u>. If $x \sim EC_n(0,I_n,\phi)$ and $p(x=0) = 0$, let

$$(6.15) \qquad \bar{x} = \frac{1}{n}\sum_1^n x_i \quad \text{and} \quad s^2 = \sum_1^n (x_i-\bar{x})^2 .$$

We can express $s^2 = x'(I_n-M)x$ with $M = \frac{1}{n}\varepsilon_n\varepsilon_n'$. It is easy to check that $(I-M)^2 = I-M$ and $\mathrm{rk}(I-M) = n-1$. By Theorems 1 and 2 $s^2 \sim G_2(\frac{n-1}{2}; \frac{1}{2}; \phi)$ and $s^2 \sim \chi^2_{n-1}$ if and only if $\phi(t) = \exp(-t/2)$, i.e. $x \sim N_n(0,I_n)$.

Further, if $x \sim EC_n(0,\Sigma,\phi)$ with $\Sigma > 0$ and $p(x=0) = 0$, and $\bar{x}$ and s^2 are defined by (6.15), by the corollary of Theorem 1 $s^2 \sim G_2(\frac{n-1}{2}; \frac{1}{2}; \phi)$ if and only if Σ is a generalized inverse of $(I-M)$. Let Γ be an orthogonal matrix with the first row $(1/\sqrt{n},\ldots,1/\sqrt{n})$, then

$$\Gamma'(I-M)\Gamma = \begin{pmatrix} 0 & 0' \\ 0 & I_{n-1} \end{pmatrix} .$$

or

$$\underset{\sim}{I}-\underset{\sim}{M} = \underset{\sim}{\Gamma}\begin{pmatrix} 0 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I}_{n-1} \end{pmatrix}\underset{\sim}{\Gamma}' .$$

It can be verified that any generalized inverse $(I-M)^-$ of $(\underset{\sim}{I}-\underset{\sim}{M})$ can be written as

$$(\underset{\sim}{I}-\underset{\sim}{M})^- = \underset{\sim}{\Gamma}\begin{pmatrix} c_{11} & \underset{\sim}{c}_{12} \\ \underset{\sim}{c}_{21} & \underset{\sim}{I}_{n-1} \end{pmatrix}\underset{\sim}{\Gamma}' ,$$

where c_{11}, $\underset{\sim}{c}_{12}$ and $\underset{\sim}{c}_{21}$ are arbitrary. Hence $\underset{\sim}{\Sigma}$ must have the above structure, but here $\underset{\sim}{\Sigma} > 0$, $\{\underset{\sim}{c}_{ij}\}$ have to satisfy the conditions: $\underset{\sim}{c}_{21} = \underset{\sim}{c}_{12}'$ and $c_{11} > \underset{\sim}{c}_{12}\underset{\sim}{c}_{21}$.

Example 2. Assume $\underset{\sim}{x} \sim EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\phi)$, $p(\underset{\sim}{x}=\underset{\sim}{0}) = 0$, and $\bar{x}$ and s^2 are defined by (6.15), we want to point out that

$$t = \sqrt{n(n-1)}\,\frac{\bar{x}}{s} \sim t_{(n-1)} . \tag{6.16}$$

Let $\underset{\sim}{\Gamma}$ be the same as Example 1 and $\underset{\sim}{y} = \underset{\sim}{\Gamma}\underset{\sim}{x}$, then $\underset{\sim}{y} \sim EC_n(\underset{\sim}{0},\underset{\sim}{I}_n,\phi)$ and $y_1 = \sqrt{n}\,\bar{x}$, $s^2 = \Sigma_2^n y_i^2$. If $\underset{\sim}{z} \sim N_n(\underset{\sim}{0},\underset{\sim}{I}_n)$, then

$$t = \frac{\sqrt{n-1}\;y_1}{\sqrt{\sum_2^n y_i^2}} = \sqrt{n-1}\;\frac{y_1/\|\underset{\sim}{y}\|}{\sqrt{\sum_2^n y_i^2/\|\underset{\sim}{y}\|^2}} \overset{d}{=} \sqrt{n-1}\;\frac{z_1/\|\underset{\sim}{z}\|}{\sqrt{\sum_2^n z_i^2/\|\underset{\sim}{z}\|^2}} = \sqrt{n-1}\;\frac{z_1}{\sqrt{\sum_2^n z_i^2}} \sim t_{(n-1)} ,$$

because $(y_1/\|\underset{\sim}{y}\|,\ldots,y_n/\|\underset{\sim}{y}\|) \overset{d}{=} (z_1/\|\underset{\sim}{z}\|,\ldots,z_n/\|\underset{\sim}{z}\|)$, hence their Borel functions have the same distribution.

Example 3. Multivariate t-distribution. If $\underset{\sim}{x} \sim N_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$, $s \sim \chi_\nu$ (i.e., $s^2 \sim \chi^2_\nu$) and $\underset{\sim}{x}$ is independent of s, the distribution of $\underset{\sim}{y} \equiv \sqrt{\nu}\,\underset{\sim}{x}/s$ is called the multivariate t-distribution. (cf. Johnson and Kotz (1972)). When $\underset{\sim}{\mu} = \underset{\sim}{0}$, $\underset{\sim}{\Sigma} = \underset{\sim}{I}_n$, the density of $\underset{\sim}{y}$ is

$$(6.17) \qquad p_{\underset{\sim}{y}}(\underset{\sim}{y}) = \frac{\Gamma(\frac{1}{2}(n+\nu))}{(\pi\nu)^{\frac{1}{2}n}\Gamma(\frac{1}{2}\nu)}\left(1+\nu^{-1}\sum_1^n y_i^2\right)^{-\frac{1}{2}(n+\nu)} .$$

It can be found that the density of $\|\underset{\sim}{y}\|$ is

$$(6.18) \qquad p_{\|\underset{\sim}{y}\|}(u) = \frac{2\,\Gamma(\frac{1}{2}(n+\nu))}{\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}\nu)\nu^{\frac{1}{2}n}}\, u^{n-1}(1+\nu^{-1}u^2)^{-\frac{1}{2}(n+\nu)} , \quad u \geq 0 .$$

Obviously the multivariate t-distribution is an elliptically contoured distribution. If $\underset{\sim}{x} \sim N_n(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$, from (1.2)

$$\underset{\sim}{x} \overset{d}{=} \underset{\sim}{\mu} + R\underset{\sim}{A}'\underset{\sim}{u}^{(n)}$$

where $R \sim \chi_n$ is independent of $\underset{\sim}{u}^{(n)}$ and $\underset{\sim}{A}'\underset{\sim}{A} = \underset{\sim}{\Sigma}$. Thus if $\underset{\sim}{\mu} = 0$

$$(6.19) \qquad \underset{\sim}{y} \overset{d}{=} \left(\sqrt{\nu}\,\frac{R}{s}\right)\underset{\sim}{A}'\underset{\sim}{u}^{(n)} ,$$

which is the stochastic representation of $\underset{\sim}{y}$. The density (6.18) of $\sqrt{\nu}\,R/s$ can be motivated by the F-distribution because $\frac{1}{n}\left(\sqrt{\nu}\,\frac{R}{s}\right)^2 = \frac{\nu}{n}\frac{R^2}{s^2} \sim F(n,\nu)$.

If $\underset{\sim}{A}$ is an $n \times n$ symmetric matrix,

$$\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y} = \nu\,\frac{\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}}{s^2} .$$

As $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x}$ is independent of s^2 and $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim \chi^2_k$ if and only if $\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A} = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$, hence $\frac{1}{k}\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y} \sim F(k,\nu)$ if and only if $\underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A} = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$.

Further if $\underset{\sim}{\Sigma} = \underset{\sim}{I}_n$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are projection matrices with $rk(\underset{\sim}{A}) = k$, $rk(\underset{\sim}{B}) = \ell$, then $\frac{1}{\nu}(\underset{\sim}{y}'\underset{\sim}{A}\underset{\sim}{y}, \underset{\sim}{y}'\underset{\sim}{B}\underset{\sim}{y})$ *is distributed according to multivariate* inverted Dirichlet distribution (cf. p. 238, Johnson and Kotz (1972)).

When $\nu = 1$ the multivariate t-distribution reduces to multivariate Cauchy distribution.

<u>Example 4</u>. Uniform distribution in the unit sphere in $\mathbb{R}^n$. If $\underset{\sim}{x}$ is distributed according to the uniform distribution in the unit sphere in $\mathbb{R}^n$, its density is

$$p_{\underset{\sim}{x}}(\underset{\sim}{x}) = \begin{cases} \dfrac{\Gamma(\frac{n+2}{2})}{\pi^{n/2}} & \text{if} \quad \sum_{1}^{n} x_i^2 \le 1 \\ 0 & \text{otherwise} . \end{cases} \tag{6.20}$$

It can be shown that the density of $\|\underset{\sim}{x}\|$ is

$$p_{\|\underset{\sim}{x}\|}(u) = nu^{n-1} \quad \text{for} \quad 0 \le u \le 1 ,$$

and the density of $\|\underset{\sim}{x}\|^2$ is

$$p_{\|\underset{\sim}{x}\|^2}(u) = \frac{1}{2} n u^{\frac{1}{2}n-1} \quad \text{for} \quad 0 \le u \le 1 .$$

In this case the density of $G_2(\frac{k}{2}; \frac{n-k}{2}; \phi)$ is (cf. (4.1))

$$\frac{n\,\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}\, x^{\frac{k}{2}-1} \int_{\sqrt{x}}^{1} r^{-(n-2)} (r^2-x)^{\frac{n-k}{2}-1} r^{n-1}\, dr$$

$$= \frac{n\,\Gamma(\frac{n}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}\, x^{\frac{k}{2}-1}\, \frac{1}{n-1} (r^2-x)^{\frac{n-k}{2}} \Big|_{\sqrt{x}}^{1}$$

$$= \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n+2-k}{2})}\, x^{\frac{k}{2}-1} (1-x)^{\frac{n+2-k}{2}-1} \,, \quad 0 \le x \le 1 \,.$$

It is the density of $B(\frac{k}{2}, \frac{n+2-k}{2})$. Similarly, $G_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_m;\phi)$ is equal to $D_m(\alpha_1,\ldots,\alpha_{m-1};\alpha_{m+1})$, where $2(\alpha_1 + \cdots + \alpha_m) = n$.

By the theorems in Section 4 we obtain the following conclusions: If $\underset{\sim}{A}$ is an $n \times n$ symmetric matrix, then $\underset{\sim}{x}'\underset{\sim}{A}\underset{\sim}{x} \sim B(\frac{k}{2}, \frac{n+2-k}{2})$ if and only if $\underset{\sim}{A}^2 = \underset{\sim}{A}$ and $rk(\underset{\sim}{A}) = k$. If $\underset{\sim}{A}_i$, $i = 1,\ldots,m-1$, are $n \times n$ symmetric matrices, then

$$(\underset{\sim}{x}'\underset{\sim}{A}_1\underset{\sim}{x},\ldots,\underset{\sim}{x}'\underset{\sim}{A}_{m-1}\underset{\sim}{x}) \sim D_m\left(\frac{\alpha_1}{2},\ldots,\frac{\alpha_{m-1}}{2};\ \frac{n+2-\alpha_1-\cdots-\alpha_{m-1}}{2}\right)$$

if and only if $\underset{\sim}{A}_i^2 = \underset{\sim}{A}_i$, $rk(\underset{\sim}{A}_i) = \alpha_i$, $i = 1,\ldots,m-1$ and $\underset{\sim}{A}_i\underset{\sim}{A}_j = 0$ for $i \neq j$.

Clearly, we can extend the above conclusions to the uniform distribution in ellipsoid.

References

Anderson, T. W. and Fang, K. T. (1982), On the theory of multivariate elliptically contoured distributions and their applications, Technical Report No. 54, ONR Contract N00014-75-C-0442, Department of Statistics, Stanford University, Stanford, CA.

Anderson, T. W. and Styan, G. P. H. (1980), Cochran's theorem, rank additivity, and tripotent matrices, Technical Report No. 43, ONR Contract N00014-75-C-0442, Department of Statistics, Stanford University, Stanford, CA.

Cambanis, S., Huang, S. and Simons, G. (1981), On the theory of elliptically contoured distributions, J. Multivariate Analysis, 11, 368-385.

Cochran, W. G. (1934), The distribution of quadratic forms in a normal system, with applications to the analysis of covariance, Proc. Cambridge Philos. Soc., 30, 178-191.

Craig, A. T. (1943), Note on the independence of certain quadratic forms, Ann. Math. Statist., 14, 195-197.

Das Gupta, S., Eaton, M. L., Olkin, I., Perlman, M., Savage, L. J., and Sobel, M. (1972), Inequalities on the probability content of convex regions for elliptically contoured distributions, Proc. Sixth Berkeley Symp. Math. Statist. Prob., 2, 241-264, University of California Press, Berkeley.

Devlin, S. J., Gnanadesikan, R., and Kettenring, J. R. (1976), Some multivariate applications to elliptical distributions, In Essays in Probability and Statistics (S. Ideka et al., Eds.), 365-395, Shinko Tsusho Co., Ltd., Tokyo.

Johnson, N. L. and Kotz, S. (1972), Distributions in Statistics: Continuous Multivariate Distributions. Wiley, New York.

Kariya, T. and Eaton, M. L. (1977), Robust tests for spherical symmetry, Ann. Statist., 5, 206-215.

Kelker, D. (1970), Distribution theory of spherical distributions and location scale parameter generalization, Sankhyā A, 32, 419-430.

Lord, R. D. (1954), The use of Hankel transform in statistics. I. General theory and examples, Biometrika, 41, 44-55.

Muirhead, B. R. (1980), The effects of elliptical distributions on some standard procedures involving correlation coefficients: A review, Multivariate Statistical Analysis, R. P. Gupta (Ed.), North-Holland Publishing Company.

Press, S. J. (1969), On serial correlation, Ann. Math. Statist., 40, 188-196.

Robinson, J. (1965), The distribution of a general quadratic form in normal variates, Austral. J. Statist., 7, 110-114.

Schoenberg, I. J. (1938), Metric spaces and completely monotone functions, Ann. Math., 39, 811-841.

TECHNICAL REPORTS

OFFICE OF NAVAL RESEARCH CONTRACT N00014-67-A-0112-0030 (NR-042-034)

1. "Confidence Limits for the Expected Value of an Arbitrary Bounded Random Variable with a Continuous Distribution Function," T. W. Anderson, October 1, 1969.

2. "Efficient Estimation of Regression Coefficients in Time Series," T. W. Anderson, October 1, 1970.

3. "Determining the Appropriate Sample Size for Confidence Limits for a Proportion," T. W. Anderson and H. Burstein, October 15, 1970.

4. "Some General Results on Time-Ordered Classification," D. V. Hinkley, July 30, 1971.

5. "Tests for Randomness of Directions against Equatorial and Bimodal Alternatives," T. W. Anderson and M. A. Stephens, August 30, 1971.

6. "Estimation of Covariance Matrices with Linear Structure and Moving Average Processes of Finite Order," T. W. Anderson, October 29, 1971.

7. "The Stationarity of an Estimated Autoregressive Process," T. W. Anderson, November 15, 1971.

8. "On the Inverse of Some Covariance Matrices of Toeplitz Type," Raul Pedro Mentz, July 12, 1972.

9. "An Asymptotic Expansion of the Distribution of "Studentized" Classification Statistics," T. W. Anderson, September 10, 1972.

10. "Asymptotic Evaluation of the Probabilities of Misclassification by Linear Discriminant Functions," T. W. Anderson, September 28, 1972.

11. "Population Mixing Models and Clustering Algorithms," Stanley L. Sclove, February 1, 1973.

12. "Asymptotic Properties and Computation of Maximum Likelihood Estimates in the Mixed Model of the Analysis of Variance," John James Miller, November 21, 1973.

13. "Maximum Likelihood Estimation in the Birth-and-Death Process," Niels Keiding, November 28, 1973.

14. "Random Orthogonal Set Functions and Stochastic Models for the Gravity Potential of the Earth," Steffen L. Lauritzen, December 27, 1973.

15. "Maximum Likelihood Estimation of Parameters of an Autoregressive Process with Moving Average Residuals and Other Covariance Matrices with Linear Structure," T. W. Anderson, December, 1973.

16. "Note on a Case-Study in Box-Jenkins Seasonal Forecasting of Time series," Steffen L. Lauritzen, April, 1974.

TECHNICAL REPORTS (continued)

17. "General Exponential Models for Discrete Observations," Steffen L. Lauritzen, May, 1974.

18. "On the Interrelationships among Sufficiency, Total Sufficiency and Some Related Concepts," Steffen L. Lauritzen, June, 1974.

19. "Statistical Inference for Multiply Truncated Power Series Distributions," T. Cacoullos, September 30, 1974.

Office of Naval Research Contract N00014-75-C-0442 (NR-042-034)

20. "Estimation by Maximum Likelihood in Autoregressive Moving Average Models in the Time and Frequency Domains," T. W. Anderson, June 1975.

21. "Asymptotic Properties of Some Estimators in Moving Average Models," Raul Pedro Mentz, September 8, 1975.

22. "On a Spectral Estimate Obtained by an Autoregressive Model Fitting," Mituaki Huzii, February 1976.

23. "Estimating Means when Some Observations are Classified by Linear Discriminant Function," Chien-Pai Han, April 1976.

24. "Panels and Time Series Analysis: Markov Chains and Autoregressive Processes," T. W. Anderson, July 1976.

25. "Repeated Measurements on Autoregressive Processes," T. W. Anderson, September 1976.

26. "The Recurrence Classification of Risk and Storage Processes," J. Michael Harrison and Sidney I. Resnick, September 1976.

27. "The Generalized Variance of a Stationary Autoregressive Process," T. W. Anderson and Raul P.Mentz, October 1976.

28. "Estimation of the Parameters of Finite Location and Scale Mixtures," Javad Behboodian, October 1976.

29. "Identification of Parameters by the Distribution of a Maximum Random Variable," T. W. Anderson and S.G. Ghurye, November 1976.

30. "Discrimination Between Stationary Guassian Processes, Large Sample Results," Will Gersch, January 1977.

31. "Principal Components in the Nonnormal Case: The Test for Sphericity," Christine M. Waternaux, October 1977.

32. "Nonnegative Definiteness of the Estimated Dispersion Matrix in a Multivariate Linear Model," F. Pukelsheim and George P.H. Styan, May 1978.

TECHNICAL REPORTS (continued)

33. "Canonical Correlations with Respect to a Complex Structure," Steen A. Andersson, July 1978.

34. "An Extremal Problem for Positive Definite Matrices," T.W. Anderson and I. Olkin, July 1978.

35. " Maximum likelihood Estimation for Vector Autoregressive Moving Average Models," T. W. Anderson, July 1978.

36. "Maximum likelihood Estimation of the Covariances of the Vector Moving Average Models in the Time and Frequency Domains," F. Ahrabi, August 1978.

37. "Efficient Estimation of a Model with an Autoregressive Signal with White Noise," Y. Hosoya, March 1979.

38. "Maximum Likelihood Estimation of the Parameters of a Multivariate Normal Distribution,"T.W. Anderson and I. Olkin, July 1979.

39. "Maximum Likelihood Estimation of the Autoregressive Coefficients and Moving Average Covariances of Vector Autoregressive Moving Average Models," Fereydoon Ahrabi, August 1979.

40. "Smoothness Priors and the Distributed Lag Estimator," Hirotugu Akaike, August, 1979.

41. "Approximating Conditional Moments of the Multivariate Normal Distribution," Joseph G. Deken, December 1979.

42. "Methods and Applications of Time Series Analysis - Part I: Regression, Trends, Smoothing, and Differencing," T.W. Anderson and N.D. Singpurwalla, July 1980.

43. "Cochran's Theorem, Rank Additivity, and Tripotent Matrices." T.W. Anderson and George P.H. Styan, August, 1980.

44. "On Generalizations of Cochran's Theorem and Projection Matrices," Akimichi Takemura, August, 1980.

45. "Existence of Maximum Likelihood Estimators in Autoregressive and Moving Average Models," T.W. Anderson and Raúl P. Mentz, Oct. 1980.

46. "Generalized Correlations in the Singular Case," Ashis Sen Gupta, November 1980.

47. "Updating a Discriminant Function on the Basis of Unclassified Data," G.J. McLachlan and S. Ganesalingam, November 1980.

48. "A New Proof of Admissibility of Tests in the Multivariate Analysis of Variance." T.W. Anderson and Akimichi Takemura, January, 1981.

49 "A-Optimality for Regression Designs," N. N. Chan, January 1981.

50. "Tests for Standardized Generalized Variances of Multivariate Normal Populations of Possibly Different Dimensions," Ashis Sen Gupta, February 1981.

51. "A Linear Theory for Noncausality," J.P. Florens and M. Mouchart, November 1981.

52. "On the Problems of Construction and Statistical Inference Associated With a Generalization of Canonical Variables," Ashis Sen Gupta, February 1982.

53. "Distributions of Quadratic Forms and Cochran's Theorem for Elliptically Contoured Distributions and Their Applications," T. W. Anderson and Kai-Tai Fang, May 1982.

54. "On the Theory of Multivariate Elliptically Contoured Distributions and Their Applications," T. W. Anderson and Kai-Tai Fang, May 1982.

55. "On Tests for Equicorrelation Coefficient and the Generalized Variance of a Standard Symmetric Multivariate Normal Distribution," Ashis Sen Gupta, May 1982.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 53	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Distributions of Quadratic Forms and Cochran's Theorem for Elliptically Contoured Distributions and Their Applications		5. TYPE OF REPORT & PERIOD COVERED Technical Report 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) T. W. Anderson and Kai-Tai Fang		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0442
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, California		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (NR-042-034)
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics and Probability Program Code 436 Arlington, Virginia 22217		12. REPORT DATE May 1982 13. NUMBER OF PAGES 31
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Beta distribution, characterizations of normality, Cochran's theorem, Dirichlet distribution, elliptically contoured distributions, quadratic forms, spherically symmetric distributions, tripotent matrices.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper extends the theory of the distributions of quadratic forms and Cochran's theorem from normal populations to elliptically contoured distributions. Some interesting applications are discussed. In addition, some properties of quadratic forms of normal distributions are shown, in fact, to characterize normality.

END

DATE